Comments on holographic Fermi surfaces

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ABSTRACT: Recently, a mechanism for the development of a fermi surface in a holographic model of large N QCD with a baryon chemical potential was proposed. We examine similar constructions to determine when this mechanism persists. We find a class of models in which it does.

Contents

## 1 Introduction

The development of gauge/gravity duality [1–4] has provided the opportunity to study strongly coupled gauge theories. One of the most beautiful aspects of this subject is that gauge theory phenomena take on a geometric character in the dual gravity picture. For example, confinement was shown in [5] to be related to the Hawking-Page transition in gravity, and chiral symmetry breaking was shown to be related to the geometry of brane embeddings in [6].

Recently, there has been interest in studying holographic systems in backgrounds with electromagnetic fields [7–11], and at finite baryon density [12–15] and combinations thereof [16].

In this note we further investigate a proposal made in [13] for the development of a fermi surface in a holographic model of large N QCD. In that paper a baryon chemical potential was added to the Sakai-Sugimoto model [6] by turning on the gauge field on the probe D8 brane. This gauge field was sourced by string ends on the D8 brane and it was found the minimum energy configuration with fixed baryon number had a sharp cutoff in the positions of the string endpoints. As the baryon number was increased, it was found that the position of the cut off moved from the interior toward the boundary of the space. This is interesting because of a combination of two factors: the position of the string ends are interpreted as fundamental quarks in the holographic picture, and the interior of the space corresponds to the infrared of the field theory whereas the boundary corresponds to

– 1 –
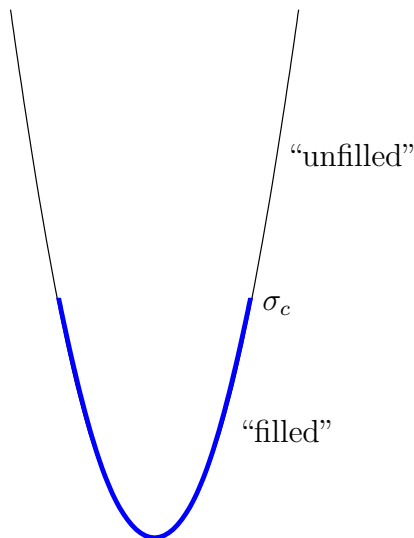
Figure 1. This sketch of the mechanism shows string endpoints occupying some region (shown in blue) below some critical point, labelled here by σ_c, which is the location of the putative fermi surface. The region below σ_c, in the interior of the space, is then considered the "filled" region and the "unfilled" region, toward the boundary of the space, is above.

the ultraviolet. This suggests that the sharp cutoff in string endpoints is a sharp cutoff in the energies of quarks, with increasing numbers of quarks corresponding to an increase in the energy of the cutoff. This sort of behaviour is what we would expect of a fermi surface for quarks, and the authors of [13] proposed that it is just that. See figure 1 for a sketch of this.

An interesting feature of this proposal is that in some sense the fermi statistics were an emergent phenomena in the holographic description. They arose from the mutual electrostatic repulsion of the string ends.

The system considered in [13] was a $3 + 1$-dimensional gauge-theory with massless fundamental quarks. In this note we investigate whether this mechanism persists in other theories. There are two sources of motivation for this. If this mechanism is peculiar feature of the Sakai-Sugimoto model, then it should not be thought of as the formation of a fermi surface, but could be a hallmark of some other interesting physics. One can, in principle, construct other brane intersections that describe strongly coupled fermions. If these systems do not exhibit this mechanism, that suggests that the mechanism is not indicating the presence of a fermi surface. Conversely, if we can show that this mechanism exists in a class of other models, then it is interesting to understand what types of examples we can construct. Since strongly coupled fermions are of interest in many areas of physics it is of interest to establish that there are diverse examples of systems that exhibit this mechanism.

In section 2 we describe the set of systems we will consider, and establish our notation and conventions. The systems include those arising from brane intersections and $\mathcal{N} = 4$ SYM on $R \times S^3$. In section 3 we will review the mechanism found in [13]. We will then look for this mechanism in a broad class of systems, and establish precise conditions for when it does and does not occur. We will find that the conditions for the non-formation of

a putitive fermi surface at finite density are extremely restrictive. The class of systems for which a putative fermi surface does develop includes strongly coupled lower dimensional systems that may be relevant for condensed matter physics. We discuss our findings in section 4.

2 Probe brane setup

In order to explore this putative mechanism for a holographic fermi surface we are interested in studying systems with fermions in the fundamental representation of the gauge group. In this work we will restrict ourselves to keeping the number of flavours fixed to one. To introduce this single flavour, we will embed a single brane in the background geometry, and work in the probe limit. In [13], the system under study was the Sakai-Sugimoto model [6] in which a probe D8-$\overline{\text{D8}}$ pair is embedded in the background of N_c D4 branes wrapped on a circle. Anti-periodic boundary conditions are taken for the fermions around the circle, as proposed in [5], so that, at energies much smaller than the Kaluza-Klein scale, the adjoint sector is pure Yang-Mills theory. The resulting theory is a $3 + 1$-dimensional gauge theory with fundamental fermions.

In this work, we are interested in understanding if the mechanism for a fermi surface proposed in [13] persists in other holographic constructions of strongly coupled gauge theories. To this end we will consider two classes of systems. The first class, including the Sakai-Sugimoto model, is constructed from D(p+1)-Dq or D(p+1)-Dq-$\overline{\text{Dq}}$ configurations. In the second class is $\mathcal{N} = 4$ SYM on $R \times S^3$ which is dual to global AdS.

2.1 D(p+1)-Dq or D(p+1)-Dq-$\overline{\text{Dq}}$ systems

In this subsection we will discuss systems that can be constructed from embedding a probe Dq brane in a D(p+1)-brane background. They are of interest in the context of looking for systems that develop fermi surfaces because they generically include light fermions that come from the string ground state in the Ramond sector. We will adopt conventions similar to those in [17–19]. The metric and dilaton for these systems have the form

$$
\begin{aligned}
ds^2 &= \left(\frac{U}{R_p}\right)^{\frac{6-p}{2}} \left(\eta_{\mu\nu}dx^\mu dx^\nu + f(U)dx_{p+1}^2\right) + \left(\frac{R_p}{U}\right)^{\frac{6-p}{2}} \left(\frac{dU^2}{f(U)} + U^2 d\Omega_{7-p}^2\right), \\
e^\phi &= \frac{g_{\text{YM}}^2}{(2\pi)^{p-1}(\alpha')^{\frac{p-2}{2}}} \left(\frac{R_p}{U}\right)^{\frac{(6-p)(2-p)}{4}},
\end{aligned}
\tag{2.1}
$$

where

$$
\begin{aligned}
f(U) &= 1 - \left(\frac{U_0}{U}\right)^{6-p}, \\
R_p^{6-p} &= g_{\text{YM}}^2 N_c d_p (\alpha')^{\frac{6-p}{2}}, \\
d_p &= 2^{5-2p} \pi^{\frac{6-3p}{2}} \Gamma\left(\frac{6-p}{2}\right).
\end{aligned}
\tag{2.2}
$$

The quantity U_0 determines the size of the compact x_{p+1} direction according to

$$\frac{R_{KK}}{U_0} = \frac{4\pi}{6-p} \left(\frac{R_p}{U_0}\right)^{\frac{6-p}{2}}. \tag{2.3}$$

It is convenient to define a new coordinate ρ according to

$$U^{\frac{6-p}{2}} = \rho^{\frac{6-p}{2}} + \frac{U_0^{6-p}}{4\rho^{\frac{6-p}{2}}}, \tag{2.4}$$

so that the metric has the form

$$ds^2 = \left(\frac{U}{R_p}\right)^{\frac{6-p}{2}} \left(\eta_{\mu\nu}dx^\mu dx^\nu + f(U)dx_{p+1}^2\right) + \left(\frac{R_p}{U}\right)^{\frac{6-p}{2}} \frac{U^2}{\rho^2} \left(d\rho^2 + \rho^2 d\Omega_{7-p}^2\right). \tag{2.5}$$

In this form it is clear that the transverse directions are conformal to flat space, and to facilitate the brane embedding, we will write these coordinates as

$$d\rho^2 + \rho^2 d\Omega_{7-p}^2 = d\lambda^2 + \lambda^2 d\Omega_l^2 + dy^2 + y^2 d\Omega_{6-l-p}^2, \tag{2.6}$$

where $\rho^2 = \lambda^2 + y^2$. In these coordinates we will take the brane to be extended in the λ and Ω_l directions and sit at a point on Ω_{6-l-p}.

We will embed a single Dq brane that intersects m of the field theory space directions, wraps $a = 0, 1$ of the Kaluza-Klein directions[1], as well as filling the λ direction and an S^l in the transverse directions. If $a = 0$ the brane configuration is of the form

$$
\begin{array}{c c c c c c c c c}
 & t & \overline{\quad p \quad} & & \mathrm{KK} & \lambda & & & \\
\mathrm{D(p{+}1)} & \times & \times \cdots \times & \times \cdots \times & \times & \bullet & \bullet \cdots \bullet & \bullet \cdots \bullet & \\
\mathrm{Dq} & \times & \times \cdots \times & \bullet \cdots \bullet & \bullet & \times & \times \cdots \times & \bullet \cdots \bullet & \\
\overline{\mathrm{Dq}} & \times & \underline{\times \cdots \times} & \bullet \cdots \bullet & \bullet & \times & \underline{\times \cdots \times} & \bullet \cdots \bullet & \\
 & & m & & & & l & &
\end{array}
\tag{2.7}
$$

and if $a = 1$ it is

$$
\begin{array}{c c c c c c c c c}
 & t & \overline{\quad p \quad} & & \mathrm{KK} & \lambda & & & \\
\mathrm{D(p{+}1)} & \times & \times \cdots \times & \times \cdots \times & \times & \bullet & \bullet \cdots \bullet & \bullet \cdots \bullet & \\
\mathrm{Dq} & \times & \underline{\times \cdots \times} & \bullet \cdots \bullet & \times & \times & \underline{\times \cdots \times} & \bullet \cdots \bullet & \\
 & & m & & & & l & &
\end{array}
\tag{2.8}
$$

The Born-Infeld action for the embedded brane is

$$S = -\mu_q \omega_l R_{KK}^a \int d^{m+a+1}x \int d\lambda H(\rho)\lambda^l \sqrt{1 + y'^2 - \frac{\rho^2}{U^2}\tilde{A}'^2}, \tag{2.9}$$

where ω_l is the volume of S^l, $\tilde{A} = 2\pi\alpha' A_0$, and

$$H(\rho) = \left(1 - \left(\frac{U_0}{U}\right)^{6-p}\right)^{\frac{a}{2}} \left(\frac{U}{\rho}\right)^{l+1} \left(\frac{U}{R_p}\right)^{\frac{(6-p)(4-\#\mathrm{ND})}{4}}, \tag{2.10}$$

---

[1]If the Dq brane does not wrap the Kaluza-Klein direction, we will assume that it sits at a fixed point.

where

$$\#\text{ND} = p + l - m - a + 2 \tag{2.11}$$

is the number of Neumann-Dirichlet directions. Systems with $a = 0$ will be D(p+1)-Dq-$\overline{\text{Dq}}$ systems and those with $a = 1$ will be D(p+1)-Dq systems.

To introduce baryons, we will use the idea of [20]. In each D(p+1) brane background is an S^{7-p} carrying flux. We will wrap D(7-p) branes on this sphere, and to satisfy the Gauss law constraint on the sphere there must be N_c strings ending on each D(7-p) brane. The other ends will sit on the Dq brane and provide a source for electric flux on the brane.

A D(7-p) brane at a position U has the Born-Infeld action

$$S = -\mu_{7-p} R_p^{6-p} \omega_{7-p} \int dt U \,. \tag{2.12}$$

The action for a collection of baryons with density $\rho_B(\lambda, x)$ is, therefore,

$$S_B = -\mu_{7-p} R_p^{6-p} \omega_{7-p} \int d^{m+a+1} x \int d\lambda U \rho_B(\lambda, x) \,. \tag{2.13}$$

In what follows we will take the baryon density to be homogeneous in the field theory space directions, $\rho_B(\lambda, x) = \rho_B(\lambda)$. There is also the effect of the string ends on the Dq brane, whereby the electromagnetic coupling contributes a term to the action of the form

$$S_s = \frac{N_c}{2\pi\alpha'} \int d^{m+a+1} x \int d\lambda \rho_B(\lambda) \tilde{A} \,. \tag{2.14}$$

2.2 Global $AdS_5 \times S^5$

In this section we will consider D7 branes probing global $AdS_5 \times S^5$. We will use the Fefferman-Graham coordinates

$$ds^2 = R^2 \left(-\frac{1}{4} \left(\frac{1}{z} + z \right)^2 dt^2 + \frac{dz^2}{z^2} + \frac{1}{4} \left(\frac{1}{z} - z \right)^2 d\Omega_3^2 + d\theta^2 + \sin^2\theta d\phi^2 + \cos^2\theta d\bar{\Omega}_3^2 \right) \tag{2.15}$$

where $z = 1$ is the centre of AdS, and the boundary is at $z = 0$.

We will then consider the D7 embedding where it fills AdS_5 as well wrapping an $S^3 \subset S^5$. The DBI action for this embedding takes the form

$$S = -\mathcal{N} \int d\Omega_3 \int d\bar{\Omega}_3 \int dt \int dz \cos^3\theta f_1(z) \sqrt{f_2(z) \left(\frac{1}{z^2} + \theta'^2 \right) - \tilde{A}'(z)^2} \tag{2.16}$$

where

$$f_1(z) = \left(\frac{1}{2} \left(\frac{1}{z} - z \right) \right)^3 \,, \qquad f_2(z) = \left(\frac{1}{2} \left(z + \frac{1}{z} \right) \right)^2 \,, \tag{2.17}$$

$\mathcal{N} = R^8 \mu_7$, and $\tilde{A} = 2\pi\alpha' A_0 / R^2$. Here $R^4 = 4\pi g_s N_c \alpha'^2$ and $\mu_7 = ((2\pi)^7 g_s \alpha'^4)^{-1}$.

Again this background carries flux, in this case through S^5. We will introduce baryons by wrapping D5 branes on this S^5. A D5 brane at position z will contribute to the action

$$S = -R^6 \mu_5 \omega_5 \int dt \frac{1}{2} \left(z + \frac{1}{z} \right) \tag{2.18}$$

The action for a collection of baryons with density $\rho_B(z, t, \Omega_3)$ is, therefore,

$$S_B = R^6 \mu_5 \omega_5 \int dt \int d\Omega_3 \int dz \frac{1}{2} \left(z + \frac{1}{z} \right) \rho_B(z, t, \Omega_3) \tag{2.19}$$

Again, we will take configurations that are homogeneous in the field theory directions so that $\rho_B(z, t, \Omega_3) = \rho_B(z)$. There is also the effect of the string endpoints that has the form

$$S_s = \frac{N_c R^2}{2\pi\alpha'} \int dt \int d\Omega_3 \int dz \tilde{A}(z) \rho_B(z) \tag{2.20}$$

3 Finding the Fermi surface

3.1 General considerations

In this section we will investigate the brane constructions we set out in the last section in the presence of baryons. We begin by noting that in all of the brane constructions we presented above, the Born-Infeld action for the probe brane took the form[2]

$$S = -\int d\sigma f(y, y', \sigma) \sqrt{1 - g(y, y', \sigma) A'^2} \,. \tag{3.1}$$

There are additional contributions to the action from the string end points and the masses of the branes

$$S' = N_c \int d\sigma \rho_B A - \int d\sigma \rho_B M(y, \sigma) \,. \tag{3.2}$$

Altogether, these give an equation of motion for the gauge field as

$$\frac{d}{d\sigma} \frac{\partial \mathcal{L}}{\partial A'} = N_c \rho_B \,. \tag{3.3}$$

It will be convenient to use the electric flux

$$E \equiv \frac{\partial \mathcal{L}}{\partial A'} = \frac{fgA'}{\sqrt{1 - gA'^2}} \,, \tag{3.4}$$

or equivalently

$$gA'^2 = \frac{E^2/g}{f^2 + E^2/g} \,, \tag{3.5}$$

which means the equation of motion for A is

$$E' = N_c \rho_B \,. \tag{3.6}$$

There will be two generic cases we would like to consider, either the probe brane extends from $\sigma = \infty$ to $\sigma = -\infty$, which we will call case 1, or ends at some $\sigma = \sigma_0$, which we will call case 2.[3] We can integrate the equation of motion for E to find that in case 1

$$2E_\infty = n_B N_c \,, \tag{3.7}$$

---

[2]We have absorbed the overall constant in f, as it will not play a role in our analysis.

[3]We are to free reparametrize σ so that $\sigma_0 = 0$ for convenience.

or in case 2

$$E_\infty = n_B N_c \,. \tag{3.8}$$

Putting this together we find

$$S' = \int d\sigma E' A - \frac{1}{N_c} \int d\sigma E' M(y,\sigma) \,, \tag{3.9}$$

and integrating by parts gives

$$S' = 2\mu E_\infty - \int d\sigma \frac{E^2/g}{\sqrt{f^2 + E^2/g}} - \frac{1}{N_c} \int d\sigma E' M(y,\sigma) \,, \tag{3.10}$$

for case 1, or

$$S' = \mu E_\infty - \int d\sigma \frac{E^2/g}{\sqrt{f^2 + E^2/g}} - \frac{1}{N_c} \int d\sigma E' M(y,\sigma) \,, \tag{3.11}$$

for case 2. Similarly, substituting A' in terms of E in S gives

$$S = - \int d\sigma \frac{f^2}{\sqrt{f^2 + E^2/g}} \,, \tag{3.12}$$

so that the total action is

$$S_{\text{total}} = - \int d\sigma \sqrt{f^2 + E^2/g} + 2\mu E_\infty - \frac{1}{N_c} \int d\sigma E' M(y,\sigma) \,, \tag{3.13}$$

for case 1, or

$$S_{\text{total}} = - \int d\sigma \sqrt{f^2 + E^2/g} + \mu E_\infty - \frac{1}{N_c} \int d\sigma E' M(y,\sigma) \,, \tag{3.14}$$

for case 2. We have eliminated the gauge field entirely from the action by expressing it in terms of the baryon charge density. We therefore find that the energy density is either

$$\mathcal{E} = \int d\sigma \sqrt{f^2 + E^2/g} - 2\mu E_\infty + \frac{1}{N_c} \int d\sigma E' M(y,\sigma) \,, \tag{3.15}$$

for case 1, or

$$\mathcal{E} = \int d\sigma \sqrt{f^2 + E^2/g} - \mu E_\infty + \frac{1}{N_c} \int d\sigma E' M(y,\sigma) \,, \tag{3.16}$$

for case 2.

The system will seek the minimum energy configuration, so we would like to minimize this energy against the baryon charge density, subject to the constraint that the overall baryon number is fixed. To ensure local minimization we vary with respect to E and find as a result that

$$\frac{E^2}{g} = \frac{f^2 g M'^2/N_c^2}{1 - g M'^2/N_c^2} \,. \tag{3.17}$$

It is this condition we will analyze to determine when a putative fermi surface will develop.

3.2 The Sakai-Sugimoto model

To understand how the putative fermi surface arises we will first consider the simplest example, the Sakai-Sugimoto model at non-zero baryon chemical potential. This has been considered previously in [13], in which the mechanism was first proposed. It will be useful to reconsider what happens in this simplest case, and to describe it in our notation before passing on to more general considerations.

The Sakai-Sugimoto model is constructed by starting with D4 branes compactified on a circle with anti-periodic (supersymmetry breaking) boundary conditions for the fermions. At low energies the theory is then weakly coupled pure Yang-Mills theory with a large number of colours N_c [5]. In the supergravity limit it is described by the background geometry

$$
ds^2 = \left(\frac{U}{R_3}\right)^{\frac{3}{2}} \left(\eta_{\mu\nu}dx^\mu dx^\nu + f(U)dx_4^2\right) + \left(\frac{R_3}{U}\right)^{\frac{3}{2}} \frac{U^2}{\lambda^2}\left(d\lambda^2 + \lambda^2 d\Omega_4^2\right),
$$
$$
e^\phi = \frac{g_{\rm YM}^2}{(2\pi)^2(\alpha')^{\frac{1}{2}}}\left(\frac{U}{R_3}\right)^{\frac{3}{4}},
$$
(3.18)

where[4]

$$
f(U) = 1 - \frac{1}{U^3},
$$
$$
U^{\frac{3}{2}} = \lambda^{\frac{3}{2}} + \frac{1}{4\lambda^{\frac{3}{2}}}.
$$
(3.19)

This geometry is what one gets by putting $p = 3$ in the general setup above. Flavour is introduced by placing D8 and $\overline{\rm D8}$ branes at the antipodal points on the compact circle, x_4. In the limit that the number of D8-$\overline{\rm D8}$ pairs N_f is much less than the number of colours, $N_f \ll N_c$, the D8-$\overline{\rm D8}$ can be treated as probing the geometry of the D4 branes and backreaction can be neglected. The action for the probe is given by

$$
S = -\mu_8\omega_4 R_3^{\frac{3}{2}}\int d^4x \int \frac{d\lambda}{\lambda}U^{\frac{7}{2}}\sqrt{1 - \frac{\lambda^2}{U^2}\tilde{A}'^2}\,.
$$
(3.20)

The field theory this setup describes at low energies, relative to the compactification scale, is a 3+1-dimensional gauge theory with only gluons in the adjoint sector and N_f flavours of fermions in the fundamental representation.

Baryons are introduced by wrapping D4 branes on the transverse S^4, and we will consider a density of baryons in the field theory directions, giving the action

$$
S_B = -\mu_4 R_3^3\omega_4\int d^4x \int d\lambda U\rho_B(\lambda)\,.
$$
(3.21)

The final contribution comes from the interaction of the string ends with the electromagnetic field on the D8 brane which gives the action

$$
S_s = \frac{N_c}{2\pi\alpha'}\int d^4x \int d\lambda \rho_B(\lambda)\tilde{A}\,.
$$
(3.22)

---

[4]We've set $U_0 = 1$ for convenience.

The various terms making up the total action are of the general form in (3.1) and (3.2) of the previous section.

In section 2 we considered a general class of systems of probe branes embedded in D-brane backgrounds. In the notation of section 2, the Sakai-Sugimoto model is a system with $p = 3$, $q = 8$, $a = 0$, $m = 3$, $l = 4$, and $\#\text{ND} = 6$. In that notation then $\rho = \lambda$, and we can set $y = y' = 0$ since it does not appear. This is because the D8 brane wraps the entire transverse S^4. Because of this, the Sakai-Sugimoto model is the simplest setting to look for this fermi surface. The form of the embedding, in terms of the functions f, g and M' given in section 3.1, is then

$$
\begin{aligned}
f(\lambda) &\propto \frac{U^{7/2}}{\lambda}\,, \\
g(\lambda) &= \frac{\lambda^2}{U^2}\,, \\
M(\lambda) &\propto U\,.
\end{aligned}
\tag{3.23}
$$

We would like to now consider the form of the electric field that minimizes the energy. It is given by (3.17) with f and g as above and

$$
M' \propto \frac{U}{\lambda} \frac{4\lambda^3 - 1}{4\lambda^3 + 1}\,.
\tag{3.24}
$$

Let us consider now what happens when λ is large. Asymptotically $f \to \infty$, $g \to 1$, and $M' \to 1$. Explicitly, this means that the form of the electric field that minimizes the energy density is

$$
E \propto \lambda^{5/2}
\tag{3.25}
$$

for large λ, and clearly grows without bound toward the boundary. However, the asymptotic value of E determines the baryon number density, which we are keeping fixed, so it is not consistent for it to grow without bound. In fact, since E must not decrease, once the electric field reaches the value E_∞ that sets the baryon number density, the best we can do to minimize the energy is to set the electric field equal to its asymptotic value. This means that there is some critical λ past which E is constant. Since in this region E is constant, then by (3.6) ρ_B must vanish in this region. However, large magnitudes for λ correspond to large energies, so that having vanishing baryon density above the critical λ means that the quarks are all below some sharp energy cutoff. This energy cutoff was proposed in [13] as the development of a quark fermi surface.

3.3 Other D(p+1)-Dq-$\overline{\text{Dq}}$ systems

The mechanism which was proposed in [13] for the formation of a Fermi surface, and reproduced above, occurred in one example among the brane constructions we have considered in 2. We would like to now understand if this mechanism can be generalized to other systems. The motivation for this is twofold. If we fail to reproduce this mechanisms in other systems with strongly coupled fermions, that would suggest that this mechanism represents some other physics than the formation of a fermi surface. Conversely, if this mechanism

can be reproduced in other systems with strongly coupled fermions, it would be suggestive that it might indeed indicate the formation of a fermi surface. In that case, it is interesting to understand in what settings it persists, and in particular to determine if there might be any that may be of relevance to condensed matter physics.

The systems we will consider can be divided into three cases, those for which, asymptotically, $y' \to 0$, $y' \to$ constant, or y' diverges. The Sakai-Sugimoto model, as considered in [13] and the previous subsection, has $y' = 0$. We will first consider systems in this class. They have the asymptotic behaviour

$$f \sim \sigma^{l + \frac{(6-p)(4-\#ND)}{4}}, \qquad g \sim 1, \qquad M' \sim 1. \tag{3.26}$$

This implies that for large σ, the electric field that minimizes the energy is given by, according to (3.17),

$$E \sim \sigma^{l + \frac{(6-p)(4-\#ND)}{4}}, \tag{3.27}$$

which diverges when $l + \frac{(6-p)(4-\#ND)}{4} > 0$. In appendix A we show that systems with $\#ND = 4$ have $y \to$ constant asymptotically, and therefore $y' \to 0$. Therefore, when these systems have $l \geq 1$ the local minimum for E diverges for large σ, and the same mechanism for the development of a fermi surface, which was found in [13] and reproduced above, occurs.

Suppose, alternatively, that $l + \frac{(6-p)(4-\#ND)}{4} \leq 0$. In the case that the inequality is strict, this would indicate that asymptotic value of the electric field vanishes. Since this asymptotic value of the electric field dictates that the baryon density must also vanish, then the system does not contain any baryons, and therefore we would not expect a fermi surface to form.

In the marginal case of $l + \frac{(6-p)(4-\#ND)}{4} = 0$ the electric field would asymptote smoothly to some finite value E_∞. This would suggest that the charge density vanishes smoothly as we approach the boundary. Though we expect that generically most of the quarks will still sit at lower energy scales their density in this case should be given by a smooth distribution that vanishes at high energies. In particular, this would indicate that a fermi surface is not forming, even though we may expect one a priori.

As a result, in the case that $y' \to 0$ asymptotically, we may only find that a putative fermi surface does not form, when we expect one to, if

$$l + \frac{(6-p)(4-\#ND)}{4} = 0. \tag{3.28}$$

This can be rewritten in a more illuminating way. In the notation we have introduced above, the fermions coming from introducing some Dq brane are confined to an m-dimensional defect in a $p+1$-dimensional gauge theory. Expressing (3.28) in terms of these parameters, only when the dimension of the defect is given by

$$m = \frac{(p-2)(7-p-q)}{2(4-p)} \tag{3.29}$$

can we not have a putative fermi surface at finite density when $y' \to 0$ asymptotically. It is straightforward to check that in the examples we are most interested in, $p = 1, 2, 3$,

that this is only possible if $m = 0$. To emphasize, in this case, only when the fermions are localized on a point-like defect is it possible for the putative fermi surface to not develop.

Some remarks are in order. As we discuss in appendix A, the asymptotic separation of the branes sets the quark mass. The above analysis showing that the putative mechanism for the fermi surface occurs was independent of the asymptotic value of the brane separation, as long as it was finite. This means that the mechanism is insensitive to the quark mass. In the system considered in [13] the quarks were massless, so we have shown this mechanism is also viable for massive quarks. We also point out that the systems for which this mechanism occurs include the $1 + 1$-dimensional D2-D4-$\overline{\text{D4}}$ system, the $2 + 1$-dimensional D3-D5-$\overline{\text{D5}}$ system, and the $3 + 1$-dimensional D4-D6-$\overline{\text{D6}}$ system. These lower dimensional systems are interesting because they may serve as useful toy models for condensed matter physics.

Next, consider the marginal case in which $y' \to$ constant asymptotically. The behaviour in this case is similar to that in the previous one; the asymptotic behaviour of the functions f and g just gain constant coefficients that depend on the asymptotic value of y', but they scale with σ in the same way. This indicates that, again, the only way that we could have the non-formation of a putative fermi surface in interesting systems is if the fermions are confined to a point-like defect.

Finally, consider the other case that y' diverges for large σ. Suppose that, for large σ, $y' \sim \sigma^k$ for some $k > 1$.[5] Asymptotically we have the behaviour

$$f \sim \sigma^{l + \frac{(6-p)(4-\#\text{ND})}{4} - k}, \qquad g \sim \frac{1}{\sigma^{2k}}, \qquad M' \sim 1. \tag{3.30}$$

This implies that the form of the electric field that minimizes the energy is, according to (3.17),

$$E \sim \sigma^{l + \frac{(6-p)(4-\#\text{ND})}{4} - 3k}, \tag{3.31}$$

which diverges when $l + \frac{(6-p)(4-\#\text{ND})}{4} > 3k$. Systems that satisfy this requirement will also exhibit the mechanism in [13] for the development of a putative fermi surface.

If we consider the in which $l + \frac{(6-p)(4-\#\text{ND})}{4} < 3k$ then the asymptotic electric field and consequently baryon density both vanish. This happened in the previous two cases as well, and it is not surprising that a fermi surface would not form under these circumstances. Only the marginal case of $l + \frac{(6-p)(4-\#\text{ND})}{4} = 3k$ is therefore interesting. Again, writing this in terms of the dimensionality of the defect, the gauge theory and the probe branes we find a putative fermi surface doesn't form only when the degree of divergence is

$$k = \frac{1}{12} \left(2m(4 - p) - (p - 2)(7 - p - q) \right). \tag{3.32}$$

In this case we would still expect that most of the quarks would sit at lower energy scales, but the distribution must vanish smoothly at higher energies.

Let us now summarize the main results of this subsection. We first considered two cases in which the embedding function y had either the asymptotic behaviour $y' \to 0$ or

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[5]In general we might also want to have some logarithmic dependence on σ as well, but this does not have any effect on our conclusions.

$y' \rightarrow$ constant. We found in these cases that the only circumstance in which we would not have the formation of a fermi surface at finite density is in a $p+1$-dimensional field theory with fermions coming from a Dq flavour brane that are localized on an m-dimensional defect where

$$m = \frac{(p-2)(7-p-q)}{2(4-p)} \, . \tag{3.33}$$

We also pointed out that for systems of interest, where $p \leq 3$, can only be satisfied by $m = 0$, i.e. by fermions localized at a point. We considered a further class of systems in which the embedding function had the asymptotic behaviour $y' \rightarrow \infty$. We found that, if the asymptotic behaviour was such that $y' \sim \sigma^k$ for some $k > 1$ in a holographic description of a $p+1$-dimensional field theory with fermions coming from a Dq flavour brane that that are localized on an m-dimensional defect, only when

$$k = \frac{1}{12} \left(2m(4-p) - (p-2)(7-p-q) \right) , \tag{3.34}$$

would a fermi surface not form at finite density. The two conditions (3.33) and (3.34) on the non-formation of a putative fermi surface at finite density are quite restrictive and indicate that generically we should expect one to form.

3.4   D(p+1)-Dq systems

The analysis in the previous section carries over almost unchanged to the case of D(p+1)-Dq systems. Note that the only modification is that the function $H(\rho)$ picks up a factor of

$$\sqrt{1 - \left(\frac{U_0}{U}\right)^{6-p}} \, , \tag{3.35}$$

as does, therefore, the function f of section 3.1. This factor asymptotes to unity, so that the conditions (3.33) and (3.34) found above hold for D(p+1)-Dq systems as well.

An interesting example system for this case is the D4-D6 system where the D6 brane intersects two of the field theory directions. This system has a $3+1$-dimensional gauge field with fundamental particles localized on a $2+1$-dimensional defect.

3.5   Global $AdS_5 \times S^5$

We would like to consider the asymptotics of this system. They have been analyzed previously in [21]. There it was shown that

$$\theta(z) \sim \theta_0 z + \theta_2 z^3 + 2\theta_0 z^3 \ln z + \cdots . \tag{3.36}$$

These asymptotics are not altered by an asymptotically constant E, which we demand to have fixed baryon number.[6] We need to also consider the asymptotics of the functions that appear in (3.17), they are

$$M' \sim -\frac{1}{2z^2} \, , \qquad g \sim 4z^4 \, , \qquad f \sim \frac{1}{16z^5} \, . \tag{3.37}$$

Together, these imply that E diverges as $z \rightarrow 0$. As before, this indicates the development of a putative fermi surface. Note the dual field theory in this case is $\mathcal{N} = 4$ SYM on $R \times S^3$, so this mechanism also works in the case of a field theory on a compact space.

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[6]It can be checked that the asymptotic value of E only enters this expansion at order $z^9 \ln z$.

4 Discussion

In this paper we have considered a broad class of holographic systems at finite baryon chemical potential. This was motivated by a proposal in [13] of a mechanism for the development of a fermi surface in a holographic model of large N QCD. We found that the mechanism for the formation of a putative fermi surface persists across a broad class of models. This suggests that the mechanism is not a peculiar feature of the Sakai-Sugimoto model that results from one of its particular features, e.g. massless quarks, it's dimensionality, the fact it's fermions are not localized on a defect, or that the embedding is simple because the flavour branes fill the transverse S^4. The systems we have studied are therefore useful for two reasons. They give some reason to believe that this mechanism may indeed describe the formation of a fermi surface, because it does occur in a wide class of systems with strongly coupled fermions at finite density. They are also interesting because they comprise a variety systems including lower dimensional ones that could find application in condensed matter physics for understanding strongly coupled phenomena in which fermi surfaces play an important role.

If the mechanism we have described in this note does indeed indicate the development of a fermi surface, the picture we have presented here is somewhat rudimentary. The theory of strongly coupled fermi liquids is a well developed subject (see, e.g. [22]), and there are general expectations for what features systems such as those under study here should exhibit. One such feature is the phenomenon of zero sound. Investigations of zero sound in holographic settings at finite density have been carried out recently in [23–25]. The zero sound modes in these cases were identified with fluctuations of the probe flavour branes, and the investigations were carried out without explicitly including a source for the baryon charge density. A fermi surface has not been identified in the absence of the sources, so this might suggest that an explicit fermi surface is not important in finding zero sound in holographic constructions. This seems confusing because zero sound is associated with deforming the fermi surface. Another way of identifying a fermi surface is by finding a pole in the retarded current-current correlation function. Such an investigation was carried out for the Sakai-Sugimoto model in [25] but that analysis failed to reveal the expected pole. However, the analysis was carried out in the absence of the putative fermi surface we considered here, and it does not seem unreasonable that if a similar analysis was carried out with it included, a sharp cutoff in the charge density might produce such a pole.

Having now established the existence of a putative fermi surface in a diverse set of examples, it would be particularly interesting to investigate what effect the presence of the putative fermi surface has on the thermodynamics and the spectrum of low energy excitations, as well as to look for a zero sound mode. It is difficult, in the general framework we have used in this paper, to address general features of the low energy spectrum for the whole class of systems under consideration, for example the existence of poles in the retarded current-current two-point function. It would interesting to revisit this question in the Sakai-Sugimoto model as was done in [25] to determine if the existence of the putative fermi surface we have described above is sufficient to produce the expected pole. Such a study is also of particular interest in some of the lower dimensional examples that we

have considered because of their potential applications for condensed matter physics. A further interesting question is to understand how the systems respond to external electric and magnetic fields. We leave these investigations to future work.

Acknowledgments

A Systems with $\#\mathrm{ND} = 4$

In this section we will consider a set of supersymmetric systems. The systems we will consider have $a = 0$ and $\#\mathrm{ND} = 4$.

First consider the situation in which the gauge field on the Dp brane is not turned on. This will allow us to determine the relationship between the asymptotics of the embedding function and the masses of the quarks and the value of the quark condensate in the gauge theory.

In this case it is convenient to make all the quantities in the action dimensionless by taking $U_0 = 1$. We would like to consider embeddings in which $y \to y_\infty$ as we take $\lambda \to \infty$. The equation of motion has the asymptotic form

$$\partial_\lambda(\lambda^l y') = \frac{l+1}{2}\lambda^{p+l-8}y\,. \tag{A.1}$$

This equation can be recast as a Bessel equation, and has the general solution

$$y(\lambda) = A\frac{1}{\lambda^{\frac{l-1}{2}}}J_{\frac{l-1}{6-p}}\left(\frac{\sqrt{2(l+1)}}{6-p}\frac{1}{\lambda^{\frac{6-p}{2}}}\right) + B\frac{1}{\lambda^{\frac{l-1}{2}}}J_{-\frac{l-1}{6-p}}\left(\frac{\sqrt{2(l+1)}}{6-p}\frac{1}{\lambda^{\frac{6-p}{2}}}\right)\,, \tag{A.2}$$

where J are the usual Bessel functions. To determine the asymptotic behaviour we note that $J_\alpha(x) \sim x^\alpha$ for small x, which means that, for $l > 1$,[7] term multiplied by A goes to a constant for $\lambda \gg 1$, and the term multiplied by B goes like $\lambda^{-(l-1)}$.

Following the analysis in [18] if we have the asymptotic behaviour

$$y(\lambda) \sim y_\infty + \frac{c}{\lambda^{l-1}} \tag{A.3}$$

then the quark mass will be

$$m_q = \frac{U_0 y_\infty}{2\pi\alpha'} \tag{A.4}$$

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[7]For $l \leq 1$ one of the solutions diverges for large λ indicating that the equation (A.1) is only valid for $l > 1$.

and the condensate will be

$$\langle \bar{\psi}\psi \rangle = -2\pi\alpha'\mu_q(l-1)\omega_l U_0^l c \,.$$ (A.5)

If we now turn on the gauge field, when $l > 1$, similar considerations give

$$\tilde{A} \sim \tilde{\mu} - \frac{\tilde{c}}{\lambda^{l-1}} \,,$$ (A.6)

where $\tilde{\mu}$ is related to the chemical potential μ by

$$\tilde{\mu} = \frac{1}{2\pi\alpha'}\mu \,,$$ (A.7)

and $\tilde{c}$ determines the number of baryons according to

$$\tilde{c} = \frac{n_B N_c}{2(l-1)\mu_q\omega_l} \,.$$ (A.8)

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